The Amplituhedron

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Work with Nima Arkani-Hamed, to appear

Motivation

Quantum field theory (QFT): main theoretical tool for description of microscopic world.

Failure to describe some of the fundamental problems of our universe: black holes, CC problem, ...

The Grand Plan: Reformulate QFT using new concepts which makes easier to do the next step.

Step 1: Try to do it for the simplest QFT – $\mathcal{N} = 4$ SYM in planar limit.

Step 1.1.: Find the reformulation for on-shell scattering amplitudes in this theory.

Introduction

Integrand of the amplitude

Object of interest: on-shell scattering amplitudes of massless states in planar $\mathcal{N}=4$ SYM at weak coupling.

Integrand of the amplitude

- In the planar limit it is an unique rational function.
- At tree-level: it is just tree-level amplitude.
- At loop-level: sum of all Feynman diagrams before integration.

$$M_n^{L-loop} = \int d^4\ell_1 \dots d^4\ell_L \;\; \mathcal{I}_n^L$$

- The integrand lives strictly in four dimensions.
- Then it must be integrated (contour, regularization, transcendental functions, etc.)

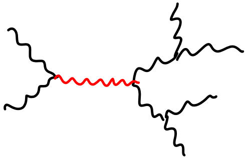
In this talk: Amplitude = Integrand of the amplitude.

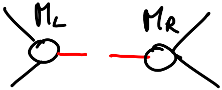
Integrand of the amplitude

Methods of calculating amplitudes

1) Feynman diagrams:

- Locality and unitarity manifest.





- Not all symmetries manifest, extremely inefficient.

2) BCFW recursion relations:

- Locality not manifest - spurious poles.
- All symmetries manifest, very efficient.

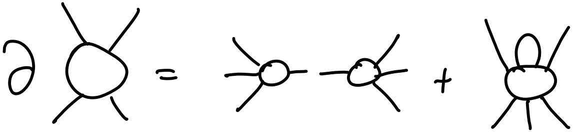
3) Wilson loop correspondence

Definition of the amplitude

What is the invariant definition of the amplitude?

Standard definition from the local QFT: Locality and Unitarity

- The amplitude is a function that has local poles and proper factorization properties
- Singularity equation:



- Feynman diagrams is a way how to make these properties manifest.
- BCFW is another way to satisfy the same equation.
- Null polygonal Wilson loops satisfy the same equation.

Positive Grassmannian

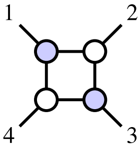
This can not be the complete story: symmetries of the theory are hidden - for planar $\mathcal{N} = 4$ SYM we have a Yangian symmetry.

New mathematical structure underlying amplitudes

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT, 1212.5605]

Remarkable relation between two different objects.

- **On-shell diagrams**: physical quantities obtained by gluing together three-point amplitudes.



- **Positive Grassmannian** $G_+(k,n)$: basic object in algebraic geometry, $k \times n$ matrix C / modulo $GL(k)$ with all $k \times k$ minors positive

Positive Grassmannian

On-shell diagrams / cells of the positive Grassmannian provide new basis of objects for the amplitude, Yangian invariant term-by-term.

Cell in the positive Grassmannian $G_+(k,n) \rightarrow$ configuration of $n$pt in $\mathbb{P}^{k-1}$, Yangian is the positive diffeomorphism on this configuration preserving positivity of $G_+(k,n)$.

For any amplitude we get a sum of these objects using recursion relations.

General framework that might be extended for large class of theories.

Recent works:

Spectral parameters for on-shell diagrams [Ferro, Lukowski, Meneghelli, Plefka, Staudacher, 1212.0850, 1308.3494]

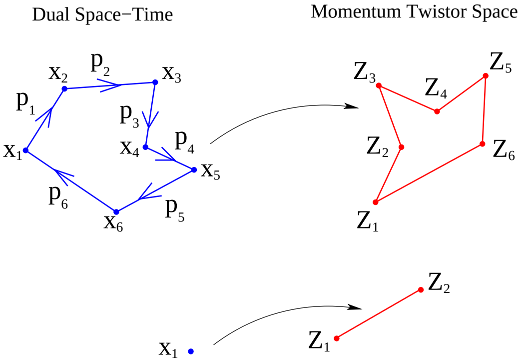
Bipartite field theories [Franco 1207.0807, 1301.0316, Franco, Galloni, Seong, 1211.5139, Franco, Uranga, 1306.6331]

Still not completely satisfactory: manifest locality is lost, the amplitude is not described as a single unique object.

Momentum twistors

[Hodges, 0905.1473]

New variables for planar theories: momentum twistors Z_i^α,



Momentum twistors

Manifest dual conformal symmetry for planar $\mathcal{N}=4$ SYM.

External particles: Z_i, η_i, loop momenta $Z_A Z_B$.

Translation between p and Z:

$$(x_i - x_j)^2 = \frac{\langle i\,i{+}1\,j\,j{+}1\rangle}{\langle i\,i{+}1\rangle\langle j\,j{+}1\rangle}, \qquad (x-x_1)^2 = \frac{\langle AB12\rangle}{\langle AB\rangle\langle 12\rangle}$$

where $\langle a\,b\,c\,d\rangle = \epsilon_{\alpha\beta\gamma\delta} Z_a^\alpha Z_b^\beta Z_c^\gamma Z_d^\delta$.

Amplitudes in planar $\mathcal{N}=4$ SYM

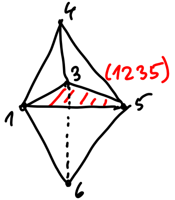
$$\mathcal{A}_{n,k} = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\cdot A_{n,k-2}(Z,\eta)$$

Polytopes

6pt NMHV split helicity amplitude $1^-2^-3^-4^+5^+6^+$:

$$A_6 = \frac{\langle 1345\rangle^3}{\langle 1234\rangle\langle 1245\rangle\langle 2345\rangle\langle 2351\rangle} + \frac{\langle 1356\rangle^3}{\langle 1235\rangle\langle 1256\rangle\langle 2356\rangle\langle 2361\rangle}$$

can be interpreted as a volume of polytope in $\mathbb{P}^3$.



Further developed for all NMHV amplitudes: polytopes in $\mathbb{P}^4$.

Vague idea:

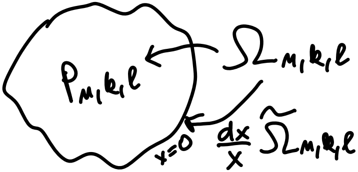
Amplitudes are "some volumes" of "some polytopes" in "some space".

We now know how to do this.

Review of the result

For each amplitude $A_{n,k}^{\ell-loop}$ we define a positive space $P_{n,k,\ell}$ - it is a generalization of the positive Grassmannian.

For each positive space we associate a form $\Omega_{n,k,\ell}$ which has logarithmic singularities on the boundary of this region.



From this form we can extract the amplitude $A_{n,k}^{\ell-loop}$.

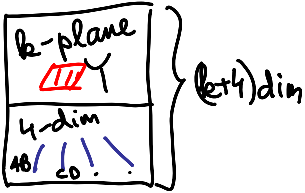
Calculating amplitudes: triangulation of the positive space $P_{n,k,\ell}$ in terms of building blocks which have trivial form.

The Amplituhedron

In the general case of n-pt L-loop N^{k}MHV amplitude we have

- Positive $k+4$-dimensional external data Z.
- k-plane Y in $k+4$ dimensions
- L lines in 4-dimensional complement to Y plane

$$\begin{aligned} Y_\sigma^I &= C_{\sigma\, a} Z_a^I \\ A_\alpha^{(1)\ I} &= C_{\alpha\, a}^{(1)} Z_a^I \\ &\vdots \\ A_\alpha^{(L)\ I} &= C_{\alpha\, a}^{(L)} Z_a^I \end{aligned} \qquad C = \begin{pmatrix} C \\ C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$



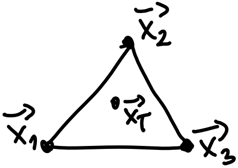
Positivity constraints:

- C is positive.
- C + any combination of $C^{(i)}$'s is positive.

The New Positive Region

Inside of the simplex

Problem from classical mechanics: center-of-mass of three points



Imagine masses c_1, c_2, c_3 in the corners.

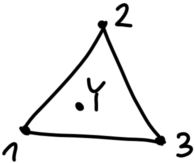
$$\vec{x_T} = \frac{c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}}{c_1 + c_2 + c_3}$$

Interior of the triangle: ranging over all positive c_1, c_2, c_3.

Triangle in projective space $\mathbb{P}^2$

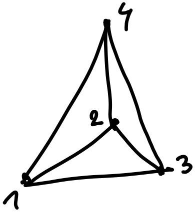
- Projective variables $Z_i = \begin{pmatrix} 1 \\ \vec{x_i} \end{pmatrix}$
- Point Y inside the triangle (mod GL(1))

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3$$



Inside of the simplex

Generalization to higher dimensions is straightforward.



Point Y inside tetrahedon in $\mathbb{P}^3$:

$$Y = c_1 Z_1 + c_2 Z_2 + c_3 Z_3 + c_4 Z_4$$

Ranging over all positive c_i spans the interior of the simplex.

In general point Y inside a simplex in $\mathbb{P}^{m-1}$:

$$Y^I = C_{1a} Z_a^I \qquad \text{where } I = 1, 2, \ldots, m$$

and C is $(1 \times m)$ matrix of positive numbers,

$$C = (c_1\, c_2\, \ldots\, c_m)/GL(1) \qquad \text{which is } G_+(1, m)$$

Into the Grassmannian

Generalization of this notion to Grassmannian

Let us imagine the same triangle and a line Y,

$$Y_1 = c_1^{(1)} Z_1 + c_2^{(1)} Z_2 + c_3^{(1)} Z_3$$
$$Y_2 = c_1^{(2)} Z_1 + c_2^{(2)} Z_2 + c_3^{(2)} Z_3$$

writing in the compact form

$$Y_\alpha^I = C_{\alpha a} Z_a^I \qquad \text{where } \alpha = 1,2$$

The matrix C is a (2×3) matrix mod GL(2) - Grassmannian $G(2,3)$.

Positivity of coefficients? No, minors are positive!

$$C = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & b \end{pmatrix}$$

Into the Grassmannian

In the general case we define a "generalized triangle"

$$Y_\alpha^I = C_{\alpha\, a} Z_a^I$$

where $\alpha = 1, 2, \ldots, k$, ie. it is a k-plane in $(k+m)$ dimensions, $a, I = 1, 2, \ldots k+m$. Simplex has $k = 1$, for triangle also $m = 2$.

The matrix C is a 'top cell' (no constraint imposed) of the positive Grassmannian $G_+(k, k+m)$, it is $k \cdot m$ dimensional.

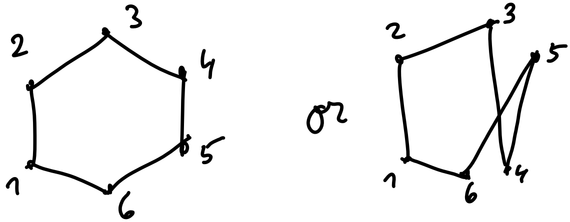
We know exactly what these matrices are!

Beyond triangles

External points Z_i did not play role, we could always choose the coordinate system such that Z is identity matrix, then $Y \sim C$.

For more vertices than the dimensionality of the space external Z's are crucial.

Let us consider the interior of the polygon in $\mathbb{P}^2$.



We need a convex polygon!

Key New Idea: Positivity of External Data

Beyond triangles

Convexity = positivity of external Z's. They form a $(3 \times n)$ matrix with all ordered minors being positive,

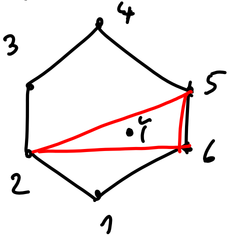
$$\langle Z_i Z_j Z_k \rangle > 0 \qquad \text{for all } i < j < k$$

The point Y inside this polygon is

$$Y = c_1 Z_1 + \cdots + c_n Z_n = C_{1a} Z_a$$

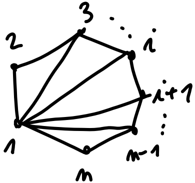
where $C \in G_+(1,n)$ and $Z \in G_+(3,n)$.

Correct but redundant description: Point Y is also inside some triangle



Beyond triangles

Triangulation: set of non-intersecting triangles that cover the region.



$$P_n = \sum_{i=2}^{n} [1\, i\, i{+}1]$$

The generic point Y is inside one of the triangles. The matrix C is

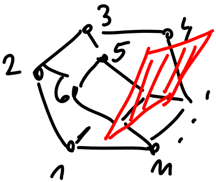
$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 & c_i & c_{i+1} & 0 & \ldots & 0 \end{pmatrix}$$

Two descriptions:

- "Top cell" $(n–1)$-dimensional of $G_+(1,n)$ - redundant.
- Collection of 2-dimensional cells of $G_+(1,n)$ - triangulation.

Into the Grassmannian

In general case:



- A k-plane Y moving in the $(k+m)$ space.
- Positive region given by n external points Z_i.
- The definition of the space:

$$Y^I_\alpha = C_{\alpha\, a} Z^I_a$$

It is a map that defines a positive region $P_{n,k,m}$,

$$G_+(k,n) \times G_+(k+m,n) \to G(k,k+m)$$

The physical case is $m = 4$.

Conjecture: The positive region $P_{n,k,4}$ represents the n-pt N^{k}MHV tree-level amplitude.

Emergent Locality and Unitarity

Locality:

- Boundary of the region: $\langle Y\, i\, j\, k\, l\rangle = 0$, ie. the k-plane Y intersects a plane $(i\, j\, k\, l)$ in the $(k+4)$ dimensional space.
- Constraints from positivity: only allowed boundaries $\langle Y\, i\, i{+}1\, j\, j{+}1\rangle$ - local poles $\sim (p_i + \cdots + p_j)^2$.

Unitarity:

- On this boundary we can show from positivity that the C matrix factorizes into positive $C^{(L)}$, $C^{(R)}$ (with an overlap).

$$C \text{ on } \langle Y\, i\, i{+}1\, j\, j{+}1\rangle = 0 \text{ becomes} \quad C \to \begin{pmatrix} C^{(L)} & 0 \\ 0 & C^{(R)} \end{pmatrix}$$

- External data also split into $Z^{(L)}$, $Z^{(R)}$ (with internal Z_I).
- On the boundary (factorization channel)

$$Y = C \cdot Z \quad \to \quad Y = C^{(L)} \cdot Z^{(L)},\;\; Y = C^{(R)} \cdot Z^{(R)}$$

Positivity implies both Locality and Unitarity.

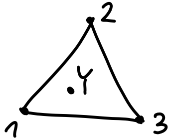
Canonical forms and amplitudes

Canonical form

How to get the actual formula from the positive region?

We define a canonical form Ω_P which has **logarithmic** singularities on the boundaries of P.

Example of triangle in $\mathbb{P}^2$:



$$\Omega_P = \frac{\langle Y\,dY\,dY\rangle\langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle}$$

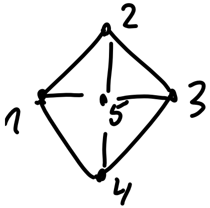
We parametrize $Y = Z_1 + c_2 Z_2 + c_3 Z_3$ and get

$$\Omega_P = \frac{dc_2}{c_2}\frac{dc_3}{c_3} = \mathrm{dlog\,c_2\ dlog\,c_3}$$

Logarithmic singularities when moving with Y on a line (12) for $c_3 = 0$ or a line (13) for $c_2 = 0$.

Canonical form

Simplex in $\mathbb{P}^4$ - this is relevant for physics.



$$\Omega_P = \frac{\langle Y\, dY\, dY\, dY\, dY\rangle\langle 12345\rangle^2}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

For $Y = Z_1 + c_2Z_2 + c_3Z_3 + c_4Z_4 + c_5Z_5$ we get

$$\Omega_P = \mathrm{dlog}\, \mathrm{c}_2\ \mathrm{dlog}\, \mathrm{c}_3\ \mathrm{dlog}\, \mathrm{c}_4\ \mathrm{dlog}\, \mathrm{c}_5$$

"Generalized triangle" given by $Y^I_\alpha = C_{\alpha\, a} Z^I_a$ with $C_{\alpha\, a} \in G_+(k, k{+}m)$.

- C is parametrized by $k \cdot m$ parameters - it is a $k \cdot m$ dimensional "top" cell of $G_+(k, k{+}m)$.
- We know all the matrices C as functions of km positive variables c_j.
- The form associated with this region is

$$\Omega_P = \mathrm{dlog}\, c_1\ \mathrm{dlog}\, c_2\ \ldots\ \mathrm{dlog}\, c_{km}$$

Canonical form

For general positive region P we have the same definition of Ω_P: canonical form with logarithmic singularities on the boundaries of P.

$$\Omega_P = \frac{\text{Measure of Y } \times \text{ Numerator}(Y, Z_i)}{\prod \langle Y \,\text{boundary}\rangle}$$

such that the form has logarithmic singularities on the boundaries.

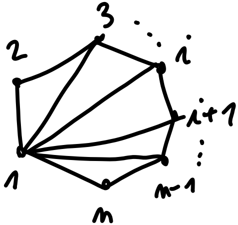
There is a natural strategy how to find the form:

- Triangulate the space, ie. find the set of non-overlapping "generalized triangles" that cover the space.
- Write the form for each triangle: dlogs of all variables $c_1, \ldots, c_{km}$.
- Solve for variables c_j in terms of Y, Z_i for each "triangle", plug into the form and sum all "triangles".

The non-trivial operation: Triangulation of the positive region!

Canonical form

Example: Polygon



$$\Omega_P = \sum_{i=2}^{n} \frac{\langle Y\, dY\, dY\rangle \langle 1\, i\, i{+}1\rangle^2}{\langle Y\, 1\, i\rangle \langle Y\, 1\, i{+}1\rangle \langle Y\, i\, i{+}1\rangle}$$

Spurious poles $\langle Y\, 1\, i\rangle$ cancel in the sum.

The positive region is not known to mathematicians (only the "triangles" which are positive Grassmannians $G_+(k,n)$).

Canonical form

The case of physical relevance is $m = 4$.

BCFW provides for us a triangulation of the space, different representations are different triangulations.

Spurious poles are internal boundaries that are absent once we put all pieces together.

Using BCFW we did many checks that the the picture is indeed correct!

We have also examples of triangulations that are not BCFW or anything else coming from physics.

From canonical forms to amplitudes

How to extract the amplitude from Ω_P?

Look at the example of simplex in $\mathbb{P}^4$.

$$\Omega_P = \frac{\langle Y\,dY\,dY\,dY\,dY\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

Note that the data are five-dimensional, it is purely bosonic and it is a form rather than function.

Let us rewrite Z_i as four-dimensional part and its complement

$$Z_i = \begin{pmatrix} z_i \\ \delta z_i \end{pmatrix} \qquad \text{where} \quad \delta z_i = (\eta_i \cdot \phi)$$

We define a reference point Y^* which is in the complement of 4d data z_i,

$$Y^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

From canonical forms to amplitudes

We integrate the form, using $\langle Y^* 1234\rangle = \langle 1234\rangle$, etc. we get

$$\int d^4\phi \int \delta(Y - Y^*)\, \Omega_P = \frac{(\langle 1234\rangle\eta_5 + \langle 2345\rangle\eta_1 + \cdots + \langle 5123\rangle\eta_4)^4}{\langle 1234\rangle\langle 2345\rangle\langle 3451\rangle\langle 4512\rangle\langle 5123\rangle}$$

For higher k we have $(k{+}4)$ dimensional external Z_i,

$$Z_i = \begin{pmatrix} z_i \\ (\eta_i\cdot\phi_1) \\ \vdots \\ (\eta_i\cdot\phi_k) \end{pmatrix} \qquad\qquad Y^* = \begin{pmatrix} \vec{0} & \vec{0} & \dots & \vec{0} \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Reference k-plane Y^* orthogonal to external z_i. We consider integral

$$A_{n,k} = \int d^4\phi_1 \dots d^4\phi_k \int \delta(Y - Y^*)\, \Omega_{P_{n,k}}$$

Loop amplitudes

MHV amplitudes

Let us start with MHV amplitudes where there is no dependence on η. External data are just original $Z_i = z_i$.

The loop variable is represented by a line $Z_A Z_B$, at one-loop we have just one line parametrized as

$$A_\alpha^I = C_{\alpha\, a} Z_a^I, \qquad \text{where } \alpha = 1, 2$$

where $A_\alpha = (A, B)$. We demand the matrix of coefficients to be positive, ie. $C \in G_+(2, n)$ and $Z \in G_+(4, n)$.

Possible boundaries of this region are generic $\langle AB\, i\, j\rangle = 0$, but only $\langle AB\, i\, i{+}1\rangle = 0$ are compatible with positivity - local poles as boundaries of the space.

MHV amplitudes

"Triangles" are just 4-dimensional cells of $G_+(2,n)$: "kermits"

Natural triangulation

$$P_n = \sum_{i<j} [1, i, i{+}1; 1, j, j{+}1]$$

where

$$C_{1,i,i+1;1,j,j+1} = \begin{pmatrix} 1 & 0 & \dots & 0 & c_i & c_{i+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & c_j & c_{j+1} & 0 & \dots & 0 \end{pmatrix}$$

Each kermit has a simple form $\Omega_P = \text{dlog}\, c_i \ \text{dlog}\, c_{i+1} \ \text{dlog}\, c_j \ \text{dlog}\, c_{j+1}$, the full MHV one-loop amplitude is then

$$\Omega_P = \sum_{i<j} \frac{\langle ABd^2A\rangle\langle ABd^2B\rangle\langle AB(i{-}1\, i\, i{+}1)\bigcap(j{-}1\, j\, j{+}1)\rangle^2}{\langle AB\,1\,i\rangle\langle AB\,1\,i{+}1\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,1\,j\rangle\langle AB\,1\,j{+}1\rangle\langle AB\,j\,j{+}1\rangle}$$

MHV amplitudes

At two-loop we have two lines $Z_A Z_B$, $Z_C Z_D$,

$$A_\alpha^{(1)\ I} = C_{\alpha\, a}^{(1)} Z_a^I$$
$$A_\alpha^{(2)\ I} = C_{\alpha\, a}^{(2)} Z_a^I$$

We combine matrices into

$$C = \begin{pmatrix} C^{(1)} \\ C^{(2)} \end{pmatrix}$$

We demand $C^{(1)}, C^{(2)}$ to be both $G_+(2,n)$. This is a "square" of one-loop problem: $(A_n^{1-loop})^2$.

Additional constraint: All (4×4) minors of C are positive! This gives MHV two-loop amplitude.

We did many numerical checks that this picture is correct.

MHV amplitudes

New feature: "triangles" are not known to mathematicians, it is a generalization of the positive Grassmannian, the form for each "triangle" is again the dlog of all positive variables.

One way to triangulate: BCFW loop recursion - we checked it triangulates the space. But geometrically it is not very natural.

New geometric triangulation for 4pt 2-loop: new formula not derivable from any physical approach.

Local expansion: not positive term by term! It is not a triangulation (perhaps some external triangulation).

MHV amplitudes

At L-loop we have L lines A_α^I.

$$A_\alpha^{(1)\ I} = C_{\alpha\, a}^{(1)} Z_a^I$$
$$\vdots$$
$$A_\alpha^{(L)\ I} = C_{\alpha\, a}^{(L)} Z_a^I$$

$$C = \begin{pmatrix} C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$

Positivity constraints:

- External data Z are positive.
- All minors of $C^{(1)}$ are positive.
- All (4×4) minors made of $C^{(i)}$, $C^{(j)}$ are positive, all (6×6) minors of $C^{(i)}$, $C^{(j)}$, $C^{(k)}$, etc. are also positive.

This conjecture passes many checks: locality, unitarity but also planarity are consequences of positivity.

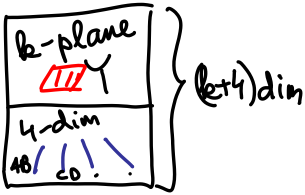
The Amplituhedron

In the general case of n-pt L-loop N^{k}MHV amplitude we have

- Positive $k+4$-dimensional external data Z.
- k-plane Y in $k+4$ dimensions
- L lines in 4-dimensional complement to Y plane

$$\begin{aligned} Y_\sigma^I &= C_{\sigma\,a} Z_a^I \\ A_\alpha^{(1)\,I} &= C_{\alpha\,a}^{(1)} Z_a^I \\ &\vdots \\ A_\alpha^{(L)\,I} &= C_{\alpha\,a}^{(L)} Z_a^I \end{aligned}$$

$$C = \begin{pmatrix} C \\ C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}$$



Positivity constraints:

- C is positive.
- C + any combination of $C^{(i)}$'s is positive.

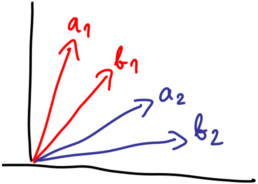
Triangulation of four-point amplitude

Four-point amplitude

The geometry problem for 4pt looks incredible simple and it should be tractable to triangulate this space to all loop orders.

We have 2d vectors a_i, b_i for $i = 1, \ldots, L$ and we demand

- They all live in the first quadrant.
- For any pair $(a_i - a_j) \cdot (b_i - b_j) < 0$.
- Triangulation: Find all possible configurations of vectors!



We do not know how to solve it in general but we have some partial results.

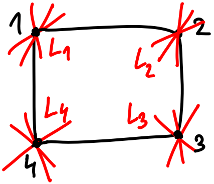
Four-point amplitude

We can triangulate manually this region up to 3-loops and give explicit result which agrees with the one in the literature.

Another direction: cuts of amplitude at all loop orders.

- The positive space is $4L$ dimensional.
- The K-cut of the amplitude represents $4L - K$ dimensional face.
- The vertices (0-dimensional faces) are leading singularities.
- We understand this object up to $2L$ dimensional faces.

Let us consider 2L cut at any loop order:



Cuts from positivity

Parametrization of C matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_i & 1 & y_i^{-1} \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -u_j^{-1} & 0 & v_j & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & -w_k^{-1} & 0 & z_k \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ -s_l & -1 & -t_l^{-1} & 0 \end{pmatrix}$$

Define

$$\Omega(x,y) = \sum_\sigma \frac{dy_1 \dots dy_\ell}{y_{\sigma_1}(y_{\sigma_2} - y_{\sigma_1}) \dots (y_{\sigma_n} - y_{\sigma_{n-1}})} \prod_i \frac{dx_i}{(x_i - y_{\sigma_n})}$$

The residue is then

$$\Omega = \Omega(v,y)\Omega(x,t)\Omega(z,u)\Omega(s,w)$$

All-loop information, impossible to get using any standard method.

Beyond the integrand

Positivity of the amplitude

The integrand itself is positive in the positive region and also integrated expressions are positive.

Positivity of 6pt MHV remainder function:

$$R_6^{2-loop} = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \frac{\pi^2 J^2}{12} + \frac{\pi^4}{72}$$

is positive in the positive region. Here the positivity implies

$$u,v,w>0, \qquad 1-u-v-w>0, \qquad \Delta = (1-u-v-w)^2 - 4uvw > 0$$

Also checked at 3-loops by Lance Dixon, also positivity of 6pt 2-loop NMHV ratio function, etc.

Towards γ_{cusp}

There is an interesting story for the logarithm of the amplitude, it is a unique object with a very special combinatorial property.

In order to approach a collinear region we need to move with all L lines. This obviously corresponds to the mild divergence.

Idea: triangulate the logarithm of the amplitude directly in this region and extract the integral for γ_{cusp}.

Introducing a spectral parameter?

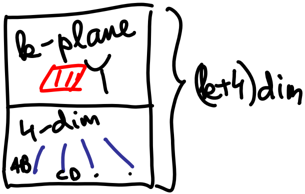
Conclusion

The Amplituhedron

The n-pt L-loop N^{k}MHV amplitude:

- Positive $k{+}4$-dimensional external data Z.
- k-plane Y in $k{+}4$ dimensions
- L lines in 4-dimensional complement to Y plane

$$
\begin{aligned}
Y^I_\sigma &= C_{\sigma\, a} Z^I_a \\
A^{(1)\ I}_\alpha &= C^{(1)}_{\alpha\, a} Z^I_a \\
&\vdots \\
A^{(L)\ I}_\alpha &= C^{(L)}_{\alpha\, a} Z^I_a
\end{aligned}
\qquad
C = \begin{pmatrix} C \\ C^{(1)} \\ \vdots \\ C^{(L)} \end{pmatrix}
$$



Positivity constraints:

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Conclusion

We defined the Positive region $P_{n,k,\ell}$ and canonical form $\Omega_{n,k,\ell}$ with logarithmic singularities on the boundaries of this region.

The n-pt ℓ-loop N^{k}MHV amplitude can be easily extracted from this form.

It is remarkable that this mathematical structure, generalizing positivity beyond the usual Positive Grassmannian, gives a complete definition of on-shell scattering amplitudes in planar $\mathcal{N}=4$ SYM.

- No reference to usual field theory notions whatsoever: no Feynman diagrams, not even on-shell diagams or recursion relations.
- Locality and Unitarity emerge from positivity.
- This rich structure is also completely new to the mathematicians.

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THANK YOU!