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XXXI. *An Electroculture Problem.* By C. Chree, *Sc.D., LL.D., F.R.S.*

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ABSTRACT.

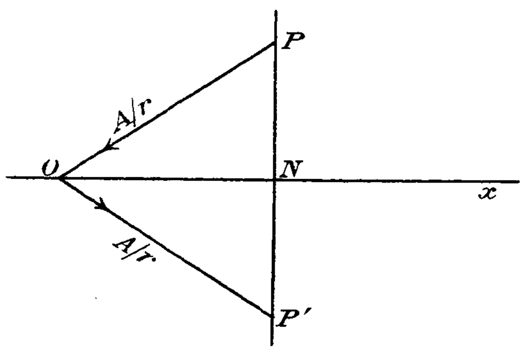
The problem is that presented by a number of fine parallel wires at the same height above the ground—treated as a plane surface—all charged to the same potential.

The chief object is to investigate how the potential gradient at ground level depends on the height and spacing of the wires. A second object is to find how the potential gradient varies with the height above the ground. The cases of a finite and an infinite number of wires are considered.

§1. The problem considered is two-dimensional, being that presented by fine parallel wires at equal intervals $2a$ apart, all at the same height h above the ground, and at the same electrical potential.

The electrostatic potential due to a single infinitely long wire in an infinite medium is $A \log_e r$, where r is the distance from the wire, and A is a constant, proportional to the electrical charge per unit length of the wire.

In practice the earth must be treated as a conductor at



zero potential. The figure is intended to represent a vertical plane perpendicular to the length of the wire, P being the section of the wire, and Ox in ground level. To maintain zero potential along Ox, we require a distribution of electricity on the ground, equivalent so far as points above ground level are concerned to a second charged wire parallel to the first, and at an equal distance from Ox. The charge per unit length of this image wire is numerically equal but of opposite sign to that in the real wire. In the figure P' represents the section of the image wire, PP' being perpendicular to Ox and bisected at N. The electrical force at O is the resultant of A/r along PO, and A/r along OP'. The resultant is thus vertical and equals $2Ah/r^2$, or $2Ah/(x^2+h^2)$, where $x=ON$.

In an electroculture area there are a large number of parallel wires, each contributing to the vertical force at ground level. Let us suppose that the charge on any one of the more remote wires instead of being concentrated on the wire is uniformly distributed over a horizontal width a on either side of the wire. If this hypothesis is made for each of the more distant wires, we have a surface charge of uniform density at height h, extending from x_0-a to x'_0+a, where x_0 and x_0' are the horizontal distances from O of the nearest and the furthest of the wires to which the hypothesis extends. The vertical force at O contributed by these more remote wires is then, according to the hypothesis, replaceable by the integral

$$\int_{x_0-a}^{x_0'+a} \frac{2Ah}{2a}\cdot\frac{dx}{h^2+x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The value is thus $(A/a)\left(\tan^{-1}\frac{x_0'+a}{h}-\tan^{-1}\frac{x_0-a}{h}\right)$, or

$$(A/a)\left(\tan^{-1}\frac{h}{x_0-a}-\tan^{-1}\frac{h}{x_0'+a}\right) \quad . \quad (2)$$

The expression inside the bracket obviously represents the angle subtended at O by the hypothetical distribution.

§2. Comprehension may be aided by discussing a particular case. Suppose, then, we have in all 21 wires, and wish to find the force under the central wire at ground level. The origin being supposed under the central wire, the horizontal distances of the 10 wires on one side are $2a$, $4a$ . . . $20a$. Each of the 21 wires has an image. The resultant vertical force at the origin due to the whole 42 wires, real and imaginary, is easily seen to be

$$\frac{2A}{h}\left[1+2\left\{\frac{1}{1+(2a/h)^2}+\frac{1}{1+(4a/n)^2}\quad . \quad . \quad +\frac{1}{1+(20a/h)^2}\right\}\right].$$

It is easy evaluating this for any convenient value of h/a. If, for instance, $h=2a$, we have only to sum

$$\frac{}{a}\left[1+2\left(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\frac{1}{26}+\frac{1}{37}+\frac{1}{50}+\frac{1}{65}+\frac{1}{82}+\frac{1}{101}\right)\right].$$

To four decimal places, this is $(A/a)\times 2{\cdot}9636$.

The contribution from the 10 most remote wires—five on either side—viz.,

$$\frac{2A}{a}\left(\frac{1}{37}+\frac{1}{50}+\frac{1}{65}+\frac{1}{82}+\frac{1}{101}\right) \text{ equals } \frac{A}{a}\times 0{\cdot}1690.$$

Suppose, however, that instead of regarding the charge as concentrated on these outside wires, we distribute it uniformly. It will then cover a patch on either side extending from $x=11a$ to $x=21a$. Taking the contributions from both patches, from the integrals we get $2(A/a)[\tan^{-1}(21a/h)-\tan^{-1}(11a/h)]$ or $2(A/a)[\tan^{-1}10{\cdot}5-\tan^{-1}5{\cdot}5]$. The angles, of course, are supposed to be in circular measure. But we have only to multiply by 0·017453 the angles as expressed in degrees in an ordinary book of logarithms, where we get

$$\tan^{-1}10{\cdot}5=84°33'{\cdot}58\equiv 84°{\cdot}560,$$

$$\tan^{-1}5{\cdot}5=79°41'{\cdot}71\equiv 79\ {\cdot}695.$$

Thus the contribution from the integral is

$$2(A/a)4{\cdot}865\times 0{\cdot}017453, \text{ or } (A/a)\times 0{\cdot}1698.$$

It is, of course, more accurate and quite as easy in this case to obtain the contribution from the 10 outside wires direct from the series as has been done above. But our immediate object is to illustrate the integration method, and show its accuracy.

The ultimate object of the integration method was really to bring the case of an infinite number of wires readily within our reach, with an accuracy sufficient for practical purposes. Suppose, then, we have obtained a result for 21 wires, and wish to find the addition to be made to this when the number becomes infinite. Taking the integral with the superior and inferior values of x as ∞ and $21a$ respectively, we have for the sum of the contributions from both sides

$$2(A/a)\{\tan^{-1}\infty-\tan^{-1}11{\cdot}5\}=(A/a)\times 0{\cdot}1899.$$

The use of the integral increases in accuracy as the inferior limit of x increases. When the inferior limit was $11a$ the error, as we have seen, was less than $\frac{1}{2}$ per cent. With an inferior limit of $21a$ the accuracy will be much better, and we may feel assured that while the result $(A/a)\times 0{\cdot}1899$ obtained above may not be accurate in the fourth decimal place, it is exact enough for all ordinary practical purposes. Adding it to the result $(A/a)(2{\cdot}9636)$ obtained above for the 21 wires, we obtain for the force at ground level directly under a wire of an infinitely extended electroculture area, having $h=2a$, the value $(A/a)(3\cdot1535)$. It is hoped that this will sufficiently explain the analytical methods employed in calculating the force at ground level. The only difference in the procedure when the force is calculated for a point midway between two wires is that the estimate is made in the first instance for the nearest 20 wires. The contribution from the more remote wires on either side when their number is infinite is derived from the result of integration, viz. $(2A/a)(\tan^{-1}\infty-\tan^{-1}20a/h)$.

§3. The vertical force could easily have been calculated at a number of different heights above ground level, but it appeared sufficient to calculate it at the half-height of the wires, and to obtain in addition its mean value between this height and ground level. The vertical force at height $h/2$ due to one wire at horizontal distance x and its image wire is

$$A\left\{\frac{h/2}{x^2+(h/2)^2}+\frac{3h/2}{x^2+(3h/2)^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Putting $x=o$, $2a$ . . . or else $x=a$, $3a$ . . . we get results applicable to a point directly under a wire or midway between two wires.

When the point is immediately under a wire the correction to allow for the difference between 21 wires and an infinite number is—

$$(A/a)(0\cdot017453)\{180-\tan^{-1}(42a/h)-\tan^{-1}(14a/h)\},$$

the angles being expressed in degrees.

For a point midway between two wires the allowance for the difference between 20 wires and an infinite number is similarly—

$$(A/a)(0\cdot017453)\{180-\tan^{-1}(40a/h)-\tan^{-1}(13\cdot\dot{3}a/h)\}.$$

§ 4. If V_0 denote the potential at height z, the vertical force, numerically considered, is $\frac{dV}{dz}$. Thus the mean value of the force between ground level and the height $h/2$ is—

$$V_{h/2}\div(h/2),$$

where $V_{h/2}$ represents the potential—which varies of course with x—at the height $h/2$.

The potential at a height $h/2$ in the vertical through the origin, due to a single wire at a horizontal distance x and its image combined, is—

$$(A/2)\log_e\frac{x^2+9h^2/4}{x^2+h^2/4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus, for the potential at height $h/2$ directly under the central wire of a series of 21 we have, when $h=2a$—

$$V_{h/2}=\tfrac{1}{2}A\log_e 10\left[\log 9+2\left\{\log\frac{13}{5}\times\frac{25}{17}\times\frac{45}{37}\times\frac{73}{65}\times\frac{109}{101}\times\frac{153}{145}\right.\right.$$
$$\left.\left.\times\frac{205}{197}\times\frac{265}{257}\times\frac{333}{325}\times\frac{409}{401}\right\}\right]. \quad (5)$$

All the logarithms inside the square bracket are to base 10. The separate factors 13/5, 25/17, &c., represent the contributions from the successive 10 wires (and their images) on each side of the central wire.

An allowance is easily made for the more remote wires. The contribution to $V_{h/2}$ from a single wire and its image as given above in (4) may be written

$$(A/2)\{\log_e(1+9h^2/4x^2)-\log_e(1+h^2/4x^2)\} \quad . \quad . \quad (6)$$

Expanding in powers of h/x, and retaining the two first terms, we get—

$$A\left(\frac{h^2}{x^2}-\frac{5}{4}\frac{h^4}{x^4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now, suppose, as before, the charge spread over a width $2a$ for each of the more remote wires, then for the contribution to the potential from the remote wires on one side of the origin we get from (7)—

$$\int_{x_0-a}^{x_0'+a}(A/2a)\left(\frac{h^2}{x^2}-\frac{5}{4}\frac{h^4}{x^4}\right)dx, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where x_0 and x_0' have the same significance as in (2).

Doubling the result of the integration, viz.—

$$\left[(A/2a)\left(\frac{-h^2}{x}+\frac{5}{12}\frac{h^4}{x^3}\right)\right]$$

taken between the limits, we get the contribution to $V_{h/2}$ from the more remote wires on both sides of the origin, and dividing the value of $V_{h/2}$ so found by $h/2$, we get the mean value of the vertical force between ground level and the height $h/2$.

As an example, suppose we have found the mean force due to the nearest 20 wires up to height $h/2$, midway between two wires, and wish to find the addition to be made when the number of wires becomes infinite. The limiting values of x in the integral are $20a$ and ∞, so its value, when doubled and divided by $h/2$ is simply

$$\frac{2A}{a}\left\{\frac{h}{20a}-\frac{5}{12}\left(\frac{h}{20a}\right)^3\right\}=\frac{Ah}{10a^2}\left\{1-\frac{1}{960}\left(\frac{h}{a}\right)^2\right\}. \quad . \quad (9)$$

Unless h/a is large, the first term gives quite a good approximation. When $h=2a$ we have $(A/a)0{\cdot}2(1-1/240)=0{\cdot}1992A/a$.

§ 5. The original investigation had two objects in view :—

1. To ascertain whether an experimental station with only a few wires could supply information likely to be of practical service.

2. To ascertain the spacing of wires appropriate to electro-culture areas.

Obviously if the wires are far apart, and near the ground, the potential gradient at ground level will be much higher at points under a wire than at points midway between wires. If a very high potential gradient were injurious, and a moderate gradient beneficial, we might have improvement of one part of the crop neutralised by injury to another. It should at least facilitate drawing conclusions, as to the merits of electro-culture, if an approach to uniformity of conditions is secured throughout the area.

In the original calculations a number of values of h/a were considered, but here results will be given mainly for three values only, viz., 1, 2 and 4, as these sufficiently illustrate the main features.

Table I. contrasts the exact results obtained from the series or the logarithms with the approximate values obtained by integration for the contribution to the force arising from the most remote 10 of a series of 21 or of 20 wires. It is confined to the values of the force at ground level, and to the mean values of the force between ground level and the half-height of the wires.

TABLE I.—*Force due to the* 10 *most remote wires* (*Coefficient of* A/a).

	Force under central wire when 21 wires.				Force midway between central wires when 20 wires.			
	At "ground level."		Mean up to $h/2$		At "ground level."		Mean up to $h/2$.	
h/a	True.	By Integral.	True.	By Integral.	True.	By Integral.	True.	By Integral.
1	0·0857	0·0861	0·0857	0·0860	0·0989	0·0991	0·0988	0·0993
2	0·1690	0·1698	0·1682	0·1689	0·1944	0·1954	0·1933	0·1942
4	0·3198	0·3211	0·3148	0·3120	0·3646	0·3662	0·3581	0·3533

The difference between the exact and approximate values for a given value of a increases, as was to be expected, with the value of h. But even for $h/a=4$ this difference is a very insignificant fraction of the force due to the nearest 21 or 20 wires. For higher values of h/a it would be an improvement to extend the exact calculation to a larger number of wires than 21 or 20, especially when considering the force or potential at heights above ground level.

The principal object of calculating Table I. was to show the degree of accuracy attainable by the approximate methods of integration.

Table II. gives the results obtained, when the number of wires is infinite, for the force arising from all more remote than the nearest 21 or 20. These results being obtained by the integration methods, are all approximate. The accuracy to be expected is considerably higher, however, than for the corresponding integration results in Table I. For a given value of a the accuracy will naturally diminish as h increases. Also the numerical value of the contribution from the more remote wires varies approximately as h when a is constant. It is roughly independent of the height above ground level, so long as the half-height of the wire is not exceeded.

TABLE II.—*Contribution to force from wires outside the central 21 or 20, when number is infinite (Coefficient of A/a).*

h/a.	Under a wire. Additional to effect of 21			Midway between wires. Additional to effect of 20		
	1	2	4	1	2	4
At "ground level" ...	0·0952	0·1899	0·3764	0·0999	0·1994	0·3948
Mean between "ground level" and $h/2$ ...	0·0951	0·1898	0·3752	0·0999	0·1992	0·3933
At $h/2$...	0·0951	0·1895	0·3732	0·0998	0·1988	0·3911

It will be observed that each numerical result in Table II. is in excess of the corresponding result in Table I., but the excess is not large.

§ 6. Table III. summarises the results obtained for the vertical force at points exactly under a wire, or exactly midway between two wires. It is convenient to express the force in terms of A/a, but it should be remembered that for a given value of h/a an increase in h means a corresponding increase in a. Increasing the height, keeping the span-height ratio constant, always entails a fall of force.

It has also to be remembered that by "ground level" in the table is really meant the surface of zero potential. It is thus to be interpreted as the top of the crop, not the actual surface of the ground in an electroculture area. If, for example, the wires over a field of wheat are 8 ft. apart and 8 ft. above the ground, a is 4 feet throughout the season; but h diminishes from 8 ft. at the start to 4 ft. at the finish, if the wheat grows to 4 ft. high. Under these circumstances, we should have h/a varying from 2 to 1, and if we had only a single wire, and maintained its potential constant, the vertical force at crop level would double in the course of the season.

TABLE III.—*Vertical force due to overhead wires (Coefficient of A/a).*

—		Directly under a wire (maximum).				Midway between wires (minimum).				Mean of max. and min.
Number of wires		1	5	21	∞	2	4	20	∞	∞
Value of h/a	Level of point.									
1	At "ground level"	2·000	3·035	3·330	3·425	2·000	2·400	2·781	2·881	3·153
	Mean up to $h/2$ ...	2·197	3·201	3·495	3·590	1·911	2·303	2·682	2·782	3·186
	At $h/2$	2·667	3·604	3·896	3·991	1·723	2·098	2·473	2·573	3·282
2	At "ground level"	1·000	2·400	2·964	3·154	1·600	2·215	2·931	3·130	3·142
	Mean up to $h/2$ ...	1·099	2·440	2·996	3·186	1·609	2·197	2·900	3·100	3·143
	At $h/2$	1·333	2·553	3·093	3·282	1·600	2·133	2·813	3·011	3·147
4	At "ground level"	0·500	1·800	2·765	3·142	0 941	1·581	2·747	3·142	3·142
	Mean up to $h/2$ ...	0·549	1·832	2·767	3·143	1·001	1·622	2·747	3·141	3·142
	At $h/2$	0·667	1·897	2·773	3·147	1·124	1·699	2·745	3·136	3·141

If the effect, whether beneficial or otherwise, depends only on the potential gradient at crop level, the "ground level" values in Table III. alone matter. If h/a is 2 or larger—*i.e.*, if the distance apart of the wires is not greater than their height—the difference, when the number of wires is infinite, between the force directly under a wire (*i.e.*, the maximum) and the force midway between two wires (*i.e.*, the minimum) is less than 1 per cent. of their mean. For practical purposes, the force may then be regarded as everywhere the same and equal to $\pi A/a$.

Even with h/a as small as 1—*i.e.*, with the height only half the distance apart of the wires—when the number of wires is infinite, the least value of the force at "ground level" is 0·81 of the largest value. With further reduction in h/a the ratio of the minimum to the maximum value falls rapidly.

For values $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{4}$ of h/a, the respective values of the ratio are only 0·68, 0·43 and 0·18 approximately.

§ 7. An infinite number of wires represents, of course, an ideal state, from which a considerable fraction of any real electroculture area must differ sensibly. As an example, suppose we travel outwards in the direction perpendicular to the wires from the centre of an electroculture area in which $h=2a$. Even at first, the value of the force at ground level is not everywhere absolutely the same, but it varies only from 3·15 A/a under a wire to 3·13 A/a midway between adjacent wires. As we approach the margin, the difference between the force under a wire and the force midway between adjacent wires increases, and, at the same time, both values fall off. When we are directly under the last wire the force has fallen to about 2·1 A/a. When we pass outside the area, travelling in the same direction, the force continually diminishes, but it remains finite for quite a considerable distance. It is still about $0{\cdot}1A/a$ when we have got to a distance of $20a$, or $10h$, from the margin.

At a considerable distance outside the area, the force for a given spacing and voltage of the wires varies approximately as the apparent height of the nearest wire. Thus, the distance to which an appreciable potential gradient effect extends increases as the height of the overhead system.

Here we have strictly confined ourselves to the electrostatic problem presented when all the charge is on the wires, and there is no appreciable excess of ions of one sign in the atmosphere. When there is appreciable free charge, with an excess of ions of one sign, the potential gradient is affected. When the overhead voltage is high enough, it must inevitably produce copious ionisation near the wires, and a strong wind will spread the ionisation to quite a long distance to leeward of the area. These and other practical considerations must be borne in mind by anyone wishing to apply the present calculations. Perhaps the most immediately useful deduction from them is that, apart from wind effects, a very uniform set of conditions can be secured at crop level if the distance between adjacent wires does not exceed the height of the wires above the crop. Also, under these conditions, the potential gradient is nearly uniform from the crop level up to at least the half height of the wires above the crop. This last point was really investigated with an eye to certain experimental investigations.

DISCUSSION.

Mr. E. V. Appleton (communicated): I presume that Dr. Chree is aware that Maxwell ("Electricity and Magnetism," Vol. I.) has dealt with the electrical theory of a charged grating of parallel wires. In Maxwell's treatment, however, the assumption is made that the distance between the wires and the conducting plane (the ground in this case) is large compared with the distance between successive wires, and in such a case it is a simple matter to deduce from Maxwell's equations that the potential gradient at the ground level is equal to $\dfrac{V}{h+\dfrac{a}{\pi}\log_e\dfrac{a}{\pi c}}$, where V is the maintained potential difference between the wires and the ground and c is the radius of the cylindrical wires. But Dr. Chree's treatment of the problem gives us information about the cases in which the distance between the wires and the ground is comparable with the distance between successive wires.

Workers on the physics of the triode vacuum tube meet the same type of problem in dealing with the action of a charged grating in producing a potential gradient near the cathode of such a tube. It has usually been assumed that Maxwell's equations give a fair approximation to the truth in the practical case when the distance between the cathode and grating is about equal to the distance between the grating wires. Dr. Chree shows us how far such an assumption is justifiable.

Mr. F. J. W. Whipple (partially communicated): I find that I was mistaken in thinking that the problem of the grating of parallel wires is one of those discussed in Sir J. J. Thomson's "Recent Researches in Electricity and Magnetism." The problem is dealt with by Maxwell himself ("Electricity and Magnetism," Vol. I., § 233). Maxwell's discussion is only directly applicable to the case in which the distance between consecutive wires is small compared with their distance from the conducting plane. The method of conjugate functions can be used effectively in working out the more general problem and leads to the following results:

If $\frac{1}{2}A$ be the line density of the charge on each of a very large number of thin parallel wires at height h above the ground, and if $2a$ be the distance between adjacent wires, then the vertical force, F, at a point on the ground at a horizontal distance, x, from one of the wires is given by the formula

$$F=\frac{\pi A}{a}\cdot\frac{\sinh\dfrac{\pi h}{a}}{\cosh\dfrac{\pi h}{a}-\cos\dfrac{\pi x}{a}}.$$

The force varies between the values

$$\frac{\pi A}{a}\coth\frac{\pi h}{2a}\quad\text{and}\quad\frac{\pi A}{a}\tanh\frac{\pi h}{2a}.$$

It will be seen that for the variation from the mean not to exceed 10 per cent. on either side $\coth\dfrac{\pi h}{2a}$ must not be greater than 1·1, and therefore h must not be less than $0{\cdot}97a$. The condition is satisfied if the height of the wires is at least half the distance between them.

It is easy to verify that the values given by Dr. Chree in Table III. for the vertical force at ground level with an infinite number of wires are in accordance with my formula.

The formula for V, the potential difference between the wires and the ground, involves r, the radius of the cross section of a wire; it is

$$V=A \cdot \log\left[\frac{2a}{\pi r}\sinh\frac{\pi h}{a}\right].$$

For comparison with the case of a single charged wire this formula may be written

$$V=A\log\frac{2h}{r}+A\log\frac{\sin \pi h/a}{\pi h/a},$$

and for comparison with the case when a uniformly charged conductor is substituted for the wires it takes the form

$$V=\frac{\pi A h}{a}+A\log\frac{a}{\pi r}+A\log\left(1-e^{-2\pi h/a}\right).$$

Dr. CHREE communicated the following reply: The experimental electro-culture area for which I first made calculations had only five wires. In any actual case only an approach is made to an infinite number. Thus a method was essential which applied to a finite number of wires. In the course of the investigations I hit on a simple method, with an obvious geometrical interpretation, which enabled me to take account of remote wires, and so to include the infinite number as a limit. I thought an exact mathematical solution for an infinite number probably existed somewhere. but having satisfied all my immediate wants I did not pursue the enquiry. Thirty years ago I read the proofs of the last edition of "Maxwell's Electricity" (and also of Sir J. J. Thomson's "Recent Researches"), but I had forgotten the existence of the solution* to which Mr. Appleton refers. The solution for an infinite number of wires should be specially useful for obtaining the exact variation of force at ground level with distance from the nearest wire in cases where a/h considerably exceeds 1. In applications of the mathematical formulæ, especially those involving the radius of the wire, and particularly to points near the wire, it should be remembered that with the usual fine wires there is a considerable sag, and there may be a corona.

* 203 *et seq.*, and Fig. XIII.