Let α, β, γ be the component displacements experienced by a particle, so that when undisturbed the coordinates are x, y, z, and when disturbed $x+\alpha$, $y+\beta$, $z+\gamma$. Then a strain of any magnitude is specified by six elements:

$$A = \left(\frac{\partial\alpha}{\partial x}+1\right)^2+\left(\frac{\partial\beta}{\partial x}\right)^2+\left(\frac{\partial\gamma}{\partial x}\right)^2 \qquad a = \frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial z}+\left(\frac{\partial\beta}{\partial y}+1\right)\frac{\partial\beta}{\partial z}+\frac{\partial\gamma}{\partial y}\left(\frac{\partial\gamma}{\partial z}+1\right)$$

$$B = \left(\frac{\partial\alpha}{\partial y}\right)^2+\left(\frac{\partial\beta}{\partial y}+1\right)^2+\left(\frac{\partial\gamma}{\partial y}\right)^2 \qquad b = \frac{\partial\alpha}{\partial z}\left(\frac{\partial\alpha}{\partial x}+1\right)+\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x}+\left(\frac{\partial\gamma}{\partial z}+1\right)\frac{\partial\gamma}{\partial x} \qquad (33)$$

$$C = \left(\frac{\partial\alpha}{\partial z}\right)^2+\left(\frac{\partial\beta}{\partial z}\right)^2+\left(\frac{\partial\gamma}{\partial z}+1\right)^2 \qquad c = \left(\frac{\partial\alpha}{\partial x}+1\right)\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\left(\frac{\partial\beta}{\partial y}+1\right)+\frac{\partial\gamma}{\partial x}\frac{\partial\gamma}{\partial y}.$$

All particles in an unstrained state, which lie on a spherical surface:

$$r_1{}^2 = \xi_1{}^2+\eta_1{}^2+\zeta_1{}^2 \qquad (34)$$

will, in a strained state, lie on an ellipsoidal surface:

$$A\xi^2+B\eta^2+C\zeta^2+2a\eta\zeta+2b\zeta\xi+2c\xi\eta = r_1{}^2. \qquad (35)$$

Accordingly, if the external forces at $P\ (x\ y\ z)$ along the axes of x, y, z, be X, Y, Z, per unit of mass, and the internal stresses be:

$$\left(\frac{d\mathfrak{p}_{xx}}{dx}+\frac{d\mathfrak{p}_{yx}}{dy}+\frac{d\mathfrak{p}_{zx}}{dz}\right)dx\,dy\,dz = \left(\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}\right)dx\,dy\,dz$$

$$\left(\frac{d\mathfrak{p}_{xy}}{dx}+\frac{d\mathfrak{p}_{yy}}{dy}+\frac{d\mathfrak{p}_{zy}}{dz}\right)dx\,dy\,dz = \left(\frac{dU}{dx}+\frac{dQ}{dy}+\frac{dS}{dz}\right)dx\,dy\,dz \qquad (36)$$

$$\left(\frac{d\mathfrak{p}_{xz}}{dx}+\frac{d\mathfrak{p}_{yz}}{dy}+\frac{d\mathfrak{p}_{zz}}{dz}\right)dx\,dy\,dz = \left(\frac{dT}{dx}+\frac{dS}{dy}+\frac{dR}{dz}\right)dx\,dy\,dz.$$

Then the equilibrium of all the forces, internal and external, leads to the following equation:

$$\left(\frac{d\mathfrak{p}_{xx}}{dx}+\frac{d\mathfrak{p}_{yx}}{dy}+\frac{d\mathfrak{p}_{zx}}{dz}+\varrho X\right)dx\,dy\,dz = \left(\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}+\varrho X\right)dx\,dy\,dz = 0$$

$$\left(\frac{d\mathfrak{p}_{xy}}{dx}+\frac{d\mathfrak{p}_{yy}}{dy}+\frac{d\mathfrak{p}_{zy}}{dz}+\varrho Y\right)dx\,dy\,dz = \left(\frac{dU}{dx}+\frac{dQ}{dy}+\frac{dS}{dz}+\varrho Y\right)dx\,dy\,dz = 0 \qquad (37)$$

$$\left(\frac{d\mathfrak{p}_{xz}}{dx}+\frac{d\mathfrak{p}_{yz}}{dy}+\frac{d\mathfrak{p}_{zz}}{dz}+\varrho Z\right)dx\,dy\,dz = \left(\frac{dT}{dx}+\frac{dS}{dy}+\frac{dR}{dz}+\varrho Z\right)dx\,dy\,dz = 0.$$

These are the general equations of equilibrium of an elastic solid, when subjected to strain by any system of forces, internal and external.

For an isotropic solid, the equations become much simplified. Using $m = k+{}^1/_3 n$, as in (31), we find the well known formulae for an elastic solid, of density ϱ per unit volume, (cf. *Thomson* and *Tait*, Treatise on Natural Philosophy, edition 1883, § 698)

$$m\frac{d}{dx}\left(\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)+n\left(\frac{\partial^2\alpha}{\partial x^2}+\frac{\partial^2\alpha}{\partial y^2}+\frac{\partial^2\alpha}{\partial z^2}\right)+\varrho X = 0$$

$$m\frac{d}{dy}\left(\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)+n\left(\frac{\partial^2\beta}{\partial x^2}+\frac{\partial^2\beta}{\partial y^2}+\frac{\partial^2\beta}{\partial z^2}\right)+\varrho Y = 0 \qquad (38)$$

$$m\frac{d}{dz}\left(\frac{\partial\alpha}{\partial x}+\frac{\partial\beta}{\partial y}+\frac{\partial\gamma}{\partial z}\right)+n\left(\frac{\partial^2\gamma}{\partial x^2}+\frac{\partial^2\gamma}{\partial y^2}+\frac{\partial^2\gamma}{\partial z^2}\right)+\varrho Z = 0.$$

When an elastic substance is strained, as in the propagation of waves, its different elements undergo changes both of form and of volume.

Let
$$\delta = \partial\alpha/\partial x+\partial\beta/\partial y+\partial\gamma/\partial z \qquad (39)$$
denote the amount of dilatation in volume experienced by an element of the substance and put
$$\nabla^2 = d^2/dx^2+d^2/dy^2+d^2/dz^2 \qquad (40)$$
for the Laplacian operation: then we shall be able to reduce these expressions (38) to the very simple form:

$$m\cdot d\delta/dx+n\nabla^2\alpha+\varrho X = 0$$
$$m\cdot d\delta/dy+n\nabla^2\beta+\varrho Y = 0 \qquad (41)$$
$$m\cdot d\delta/dz+n\nabla^2\gamma+\varrho Z = 0.$$

Now when the solid is isotropic, the density may be omitted in these formulae, or taken as unity. Accordingly if we differentiate these successive equations with respect to x, y, z respectively, and add the results, we shall get the equation for an Isotropic Solid:

$$(m+n)\nabla^2\delta+(dX/dx+dY/dy+dZ/dz) = 0. \qquad (42)$$

II. Identity of the Dilatation $\delta = \partial\alpha/\partial x+\partial\beta/\partial y+\partial\gamma/\partial z$ with the Potential V for an Infinite Elastic Solid: Confirmation of the Wave-Theory by Lord *Kelvin*'s Integrals of 1848.

It is remarkable that under certain conditions, to be more fully discussed hereafter, the equations of an infinite elastic solid admit of a very simple interpretation. This amounts to admitting the identity of the dilatation δ with the potential V, in the case of an infinite elastic solid. Indeed it was upon this tacit assumption, seventy two years ago, that Lord *Kelvin* obtained his celebrated integrals for